12.4 The Cross Product

We define the <u>cross product</u>, or <u>vector product</u>, for two 3-dimensional vectors,
$\boldsymbol{a} = \langle a_1, a_2, a_3\rangle$ and
$\boldsymbol{b} = \langle b_1, b_2, b_3\rangle$,
by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

$$(a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$$

Ex: $\boldsymbol{a} = \langle 1,2,0\rangle$ and $\boldsymbol{b} = \langle -1,3,2\rangle$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} =$$

$$(2\cdot 2 - 0\cdot 3)\boldsymbol{i} - (1\cdot 2 - 0\cdot(-1))\boldsymbol{j} + (1\cdot 3 - 2\cdot(-1))\boldsymbol{k}$$

$$\boxed{\langle 4, -2, 5\rangle}$$

NOTE:

$\langle 4, -2, 5\rangle \cdot \langle 1, 2, 0\rangle = 4 - 4 + 0$
$= 0$ ☆

$\langle 4, -2, 5\rangle \cdot \langle -1, 3, 2\rangle = -4 - 6 + 10$
$= 0$ ☆

You do: $\boldsymbol{a} = \langle 1,3,-1\rangle$, $\boldsymbol{b} = \langle 2,1,5\rangle$.

Compute $\boldsymbol{a} \times \boldsymbol{b}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ 2 & 1 & 5 \end{vmatrix}$$

$$= (15 - -1)\vec{i} - (5 - -2)\vec{j} + (1 - 6)\vec{k}$$

$$= \langle 16, -7, -5\rangle$$

NOTE: $\langle 16,-7,-5\rangle \cdot \langle 1,3,-1\rangle = 16-21+5 = 0$ ✓

$\langle 16,-7,-5\rangle \cdot \langle 2,1,5\rangle = 32-7-25 = 0$ ✓

Most important fact:
The vector $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to *both* **a** and **b**.

$\boxed{\text{proof}}$

$$\langle \underset{1}{a_2 b_3} - \underset{4}{a_3 b_2},\ a_3 b_1 - \underset{2}{a_1 b_3},\ \underset{5}{a_1 b_2}\ ??\ \rangle \cdot \langle a_1, a_2, a_3 \rangle = 0$$

$$\langle \underset{1}{a_2 b_3} - \underset{4}{a_3 b_2},\ a_3 b_1 - a_1 b_3,\ \underset{3}{-a_2 b_1} \rangle \cdot \langle b_1, b_2, b_3 \rangle = 0$$

$$a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_1 a_2 b_3 + a_1 a_3 b_2 - a_2 a_3 b_1 = 0$$

$$a_2 b_1 b_3 - a_3 b_1 b_2 + a_3 b_1 b_2 - a_1 b_2 b_3 + a_1 b_2 b_3 - a_2 b_1 b_3 = 0$$

So you roughly see how someone could arrive at such a formula.

Note: If **a** and **b** are parallel to each other, then there are many vectors perpendicular to both **a** and **b**.
So what happens to $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$?

Example: Give me any two vectors that are parallel and let's see.

$\vec{a} = \langle 1, -3, 4 \rangle$ $\cdot 2$

$\vec{b} = \langle 2, -6, 8 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 4 \\ 2 & -6 & 8 \end{vmatrix}$$

$= (-24 - -24)\vec{i} - (8 - 8)\vec{j} + (-6 - -6)\vec{k}$

$= \langle 0, 0, 0 \rangle$

$\boxed{\begin{array}{l} \vec{a} \times \vec{b} = \langle 0,0,0 \rangle = \vec{0} \\ \Longleftrightarrow \ \vec{a} \text{ AND } \vec{b} \text{ ARE PARALLEL} \end{array}}$

$\boxed{\begin{array}{l} \vec{a} \cdot \vec{b} = 0 \\ \Longleftrightarrow \ \vec{a} \text{ AND } \vec{b} \text{ ARE ORTHOGONAL} \end{array}}$

Right-hand rule

If the fingers of the right-hand curl from **a** to **b**, then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

$\boxed{\text{ORDER MATTERS}}$

$\langle 1, 2, 0\rangle \times \langle -1, 3, 2\rangle = \langle 4, -2, 5\rangle$

$\langle -1, 3, 2\rangle \times \langle 1, 2, 0\rangle = \langle -4, 2, -5\rangle$

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

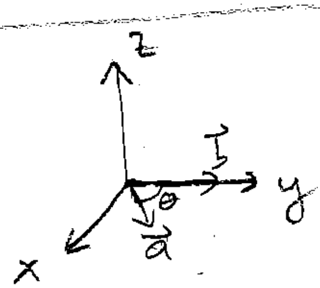
opposite direction!

$\boxed{\text{Example 1}}$

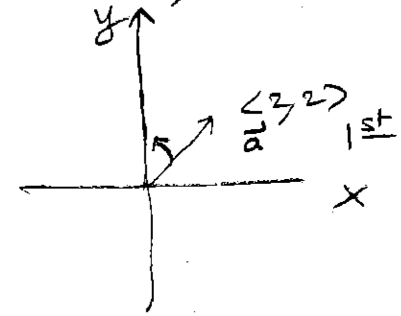
$\vec{a} = \langle 2, 2, 0\rangle$

$\vec{b} = \langle 0, 4, 0\rangle$

WILL $\vec{a} \times \vec{b}$ point upward or downward?



$\vec{a} \times \vec{b}$ WILL BE $\underline{\text{UPWARD}}$!



CHECK:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ 0 & 4 & 0 \end{vmatrix} = (0 - 0)\vec{i} - (0 - 0)\vec{j} + (8 - 0)\vec{k}$$

$= \langle 0, 0, 8\rangle$

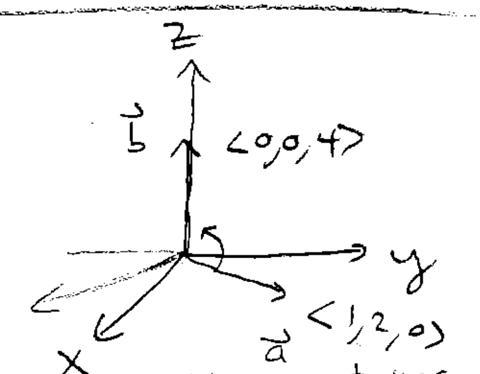
$+ \Rightarrow \boxed{\text{UPWARD}}$

$\vec{b} \times \vec{a}$ WOULD BE DOWNWARD $\langle 0, 0, -8\rangle$

EXAMPLE

$\vec{a} = \langle 1, 2, 0\rangle$

$\vec{b} = \langle 0, 0, 4\rangle$



$$\vec{a} \times \vec{b} = \langle \underbrace{+}_{\text{(+) or }-}, \underbrace{-}_{+\text{ or }(-)}, \underbrace{0}_{+\text{ or }-} \rangle$$

toward us, toward left, not up or down

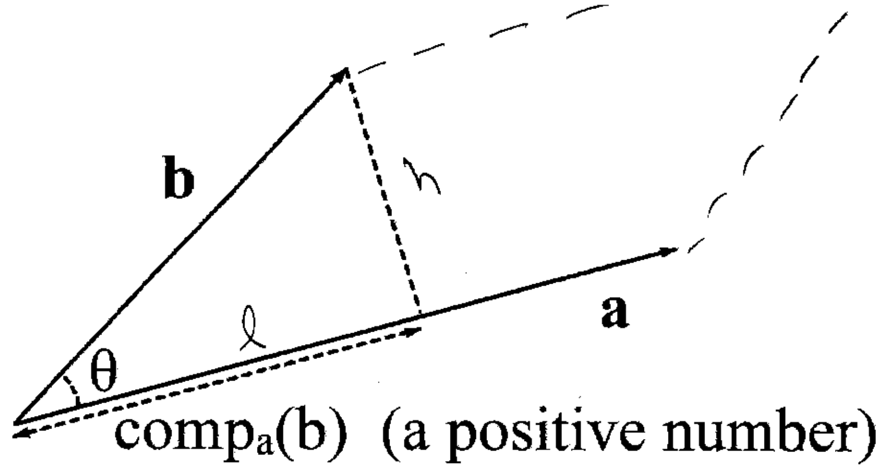
CHECK

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{vmatrix} = \langle 8, -8, 0\rangle \checkmark$$

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$*:*
Through some algebra and using the dot product rules, it can be shown that

$$|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$$

where θ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \le \theta \le \pi)$



$$\cos\theta = \frac{\ell}{|\vec{b}|}$$

$$\ell = |\vec{b}|\cos\theta$$

$$= |\vec{b}| \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\ell = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$\underbrace{\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

REVIEW
(LAST TIME

$$\sin\theta = \frac{h}{|\vec{b}|}$$

$$h = |\vec{b}|\sin\theta$$

NOTE:
$\theta = 90°$
$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|$
RECTANGLE!

$\theta = 0$ or $\theta = 180°$
$\Rightarrow |\vec{a}\times\vec{b}| = 0$
$\vec{a}\times\vec{b} = \langle 0, 0, 0\rangle$

Note: $|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$ is the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$

AREA OF TRIANGLE
FORMED BY $\vec{a}$ AND $\vec{b}$

$$= \frac{1}{2}|\vec{a}\times\vec{b}|.$$

Ex

FIND THE AREA OF THE TRIANGLE FORMED BY

$P(3,0,3)$, $Q(-2,1,4)$, $R(7,2,5)$

$\langle 0,14,-14\rangle \cdot \langle -5,1,1\rangle = 0$ ✓

$\langle 0,14,-14\rangle \cdot \langle 4,2,2\rangle = 0$ ✓

CHECK!

$\overrightarrow{PQ} = \langle -5, 1, 1\rangle$

$\overrightarrow{PR} = \langle 4, 2, 2\rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 1 & 1 \\ 4 & 2 & 2 \end{vmatrix} = (2-2)\vec{i} - (-10-4)\vec{j} + (-10-4)\vec{k}$$

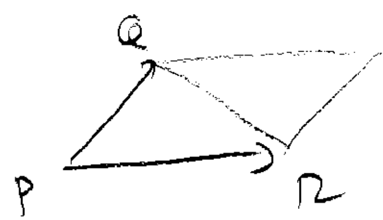
$$= \langle 0, 14, -14\rangle$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{0^2+14^2+(-14)^2} = \sqrt{2\cdot 14^2} = \boxed{14\sqrt{2}} \approx 19.79899$$

AREA OF PARALLELOGRAM

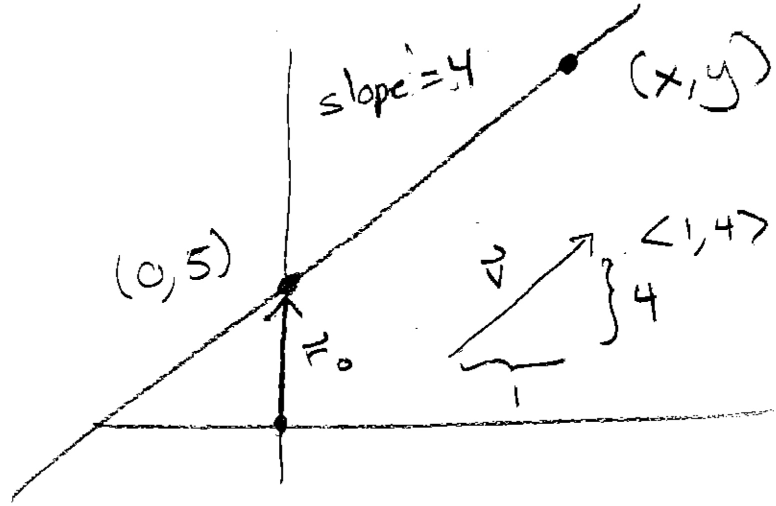
AREA OF TRIANGLE $= \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$$= \frac{1}{2}(14\sqrt{2}) = \boxed{7\sqrt{2}}$$



12.5 Intro to Lines in 3D

To describe 3D lines we use parametric equations.



Here is a 2D example

Consider the 2D line: y = 4x + 5.

(a) Find a vector parallel to the line. Call it vector **v.**

$\vec{v} = \langle 1, 4 \rangle$ works

(b) Find a vector whose head touches some point on the line when drawn from the origin. Call it vector $\mathbf{r_0}$.

$\vec{r}_0 = \langle 0, 5 \rangle$ works

(c) We can reach all other points on the line by walking along $\mathbf{r_0}$, then adding scale multiples of **v**.

some multiple

$$\underbrace{\langle x, y \rangle}_{\vec{r}} = \langle 0, 5 \rangle + t \langle 1, 4 \rangle$$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\boxed{\begin{aligned} x &= 0 + t \\ y &= 5 + 4t \end{aligned}}$$

This same idea works to describe any line in 2- or 3-dimensions.

The equation for a line in 3D:

$\boldsymbol{v} = \langle a, b, c \rangle$ = parallel to the line.

$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ = position vector

then all other points, (x, y, z), satisfy

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \mathrm{t}\langle a, b, c \rangle$,

for some number t.

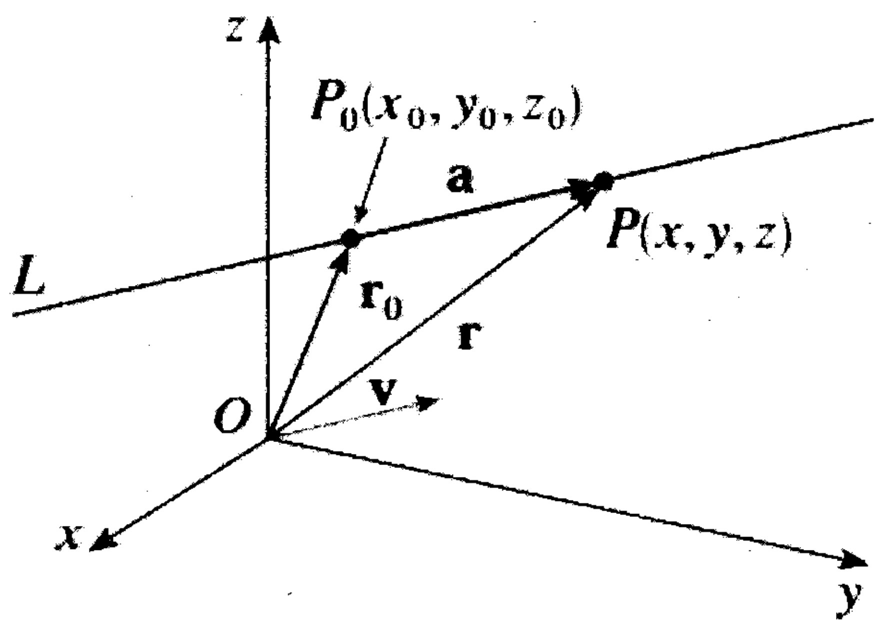
The above form ($\boldsymbol{r} = \boldsymbol{r_0} + \mathrm{t}\,\boldsymbol{v}$) is called the *vector form* of the line.

We also can write this in *parametric form* as:

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct.$$

or in *symmetric form*:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$



Basic Example – Given Two Points:
Find parametric equations of the line thru P(3, 0, 2) and Q(-1, 2, 7).

NEED

1. POSITION VECTOR : $\vec{r}_0 = \langle 3, 0, 2 \rangle$ (COULD ALSO HAVE USED $\langle -1, 2, 7 \rangle$)

2. DIRECTION VECTOR : $\vec{V} = \overrightarrow{PQ} = \langle -4, 2, 5 \rangle$ (COULD ALSO HAVE USED $\overrightarrow{QP}$)

$$\langle x, y, z \rangle = \langle 3, 0, 2 \rangle + t \langle -4, 2, 5 \rangle$$

$$\Rightarrow \boxed{\begin{aligned} x &= 3 - 4t \\ y &= 0 + 2t \\ z &= 2 + 5t \end{aligned}}$$

EVERY POINT (x, y, z) ON THE LINE SATISFIES THIS EQUATION FOR SOME VALUE OF t

$$t = \frac{x-3}{-4} \quad \text{AND} \quad t = \frac{y-0}{2} \quad \text{AND} \quad t = \frac{z-2}{5}$$

THUS,

$$\frac{x-3}{-4} = \frac{y-0}{2} = \frac{z-2}{5}$$

SIMULTANEOUSLY, IF (x, y, z) IS ON THE LINE

General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

L1: $x = 3 + 2t$, $y = -7t$, $z = 10 + t$

L2: $x = 14 + 6t$, $y = 3 - 21t$, $z = 18 + 3t$

PARALLEL

$\langle 2, -7, 1 \rangle$, $\langle 6, -21, 3 \rangle$ PARALLEL

2. Two lines **intersect** if they have an (x, y, z) point in common (use different parameters when you combine!)
   Note: The *acute angle of intersection* is the acute angle between the direction vectors.

L1: $x = t$, $y = 1 + 2t$, $z = 2 + 3t$

L2: $x = 3 - 4u$, $y = 2 - 3u$, $z = 1 + 2u$

① $t = x \stackrel{?}{=} 3 - 4u \iff t = 3 - 4u$

② $1 + 2t = y \stackrel{?}{=} 2 - 3u \Rightarrow 1 + 2(3 - 4u) \stackrel{?}{=} 2 - 3u$

$7 - 8u = 2 - 3u$

$5 = 5u$

$u = 1$

$t = -1$

③ $2 + 3t = z \stackrel{?}{=} 1 + 2u$

$-1 \neq 3$

$\boxed{\text{NO, DO NOT INTERSECT}}$

3. Two lines are **skew** if they don't intersect and aren't parallel.